Value Iteration with Guessing for Markov Chains and Markov Decisions Processes

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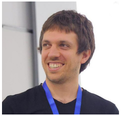

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TACAS 2025, Hamilton, Canada

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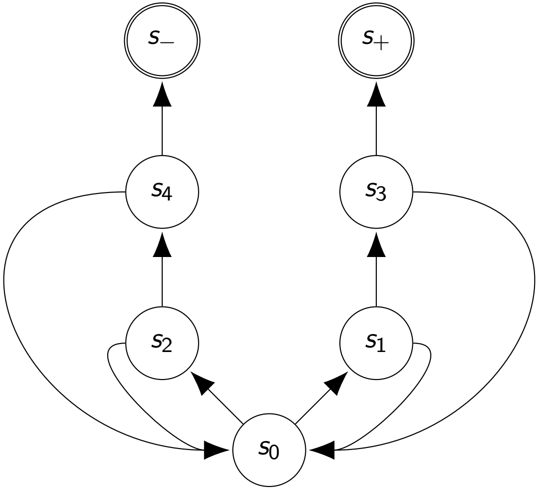
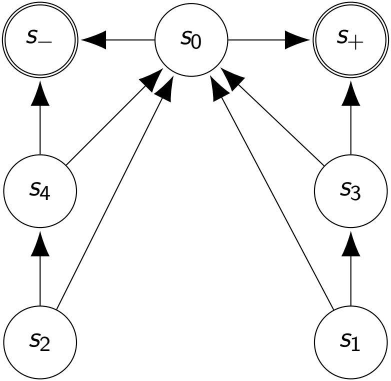
Incorporating Binary Search into Value Iteration in MDPs:

Practical implementation

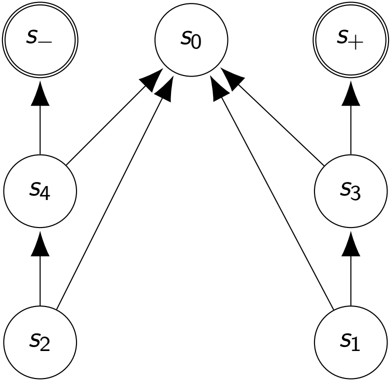
Stochastic system



Stochastic system, after guessing



Stochastic system, after guessing



Verifying a guess

Lemma

Consider a game G, a state $s \in S$ and a guess $\gamma \in [0,1]$.

$$\mathrm{Update}(s, (\gamma, \mathsf{val}_{G[s=\gamma]})) > \gamma \quad \Leftrightarrow \quad \mathsf{val}_G(s) > \gamma \,.$$

Approximately verifying a guess

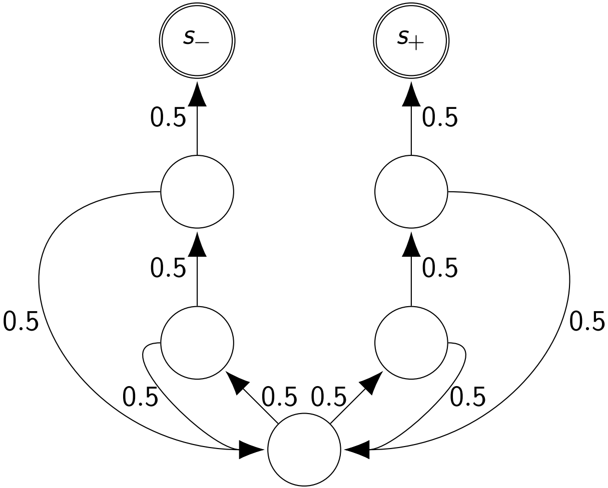
Lemma

Consider a game G, a state $s \in S$, a guess $\gamma \in [0,1]$, and $\varepsilon > 0$.

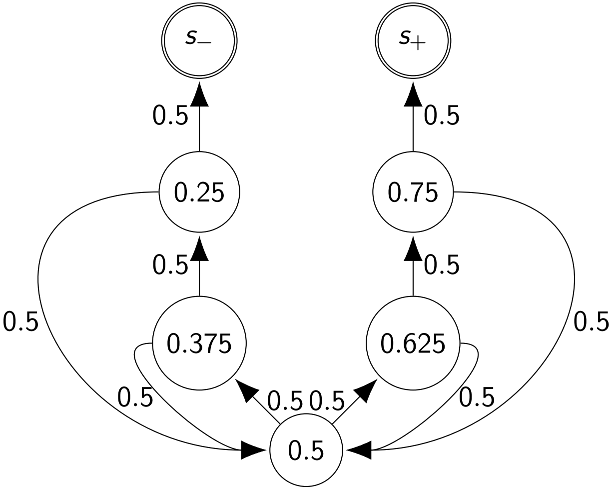
$$\mathrm{Update}(s, (\gamma, \mathsf{val}_{G[s=\gamma]})) + p_{\min}^{|S|}\,\varepsilon > \gamma \quad \Rightarrow \quad \mathsf{val}_G(s) > \gamma - \varepsilon\,.$$

Can we speed up Value Iteration?

Markov Chain



Reachability values



Update operator

For $v \in \mathbb{R}^S$,

$$\mathrm{Update}(v)(s) := \max_{a \in A} \left\{ \sum_{s' \in S} p(s' \mid s, a)\, v(s') \right\}$$

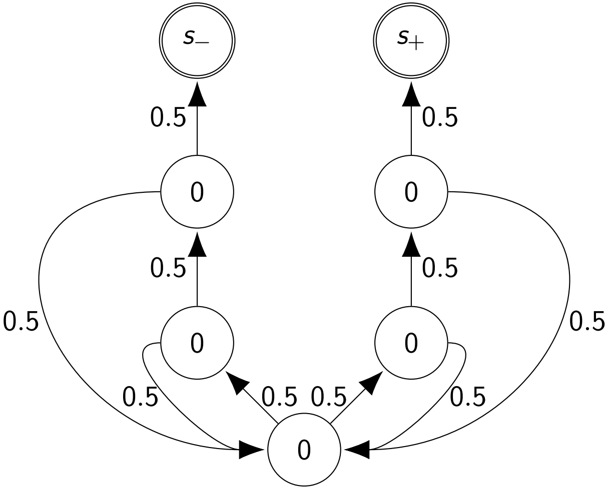
The reachability value is the least solution with

$$\mathsf{val}(s_+) = 1 \qquad\qquad \mathsf{val} = \mathrm{Update}(\mathsf{val})$$

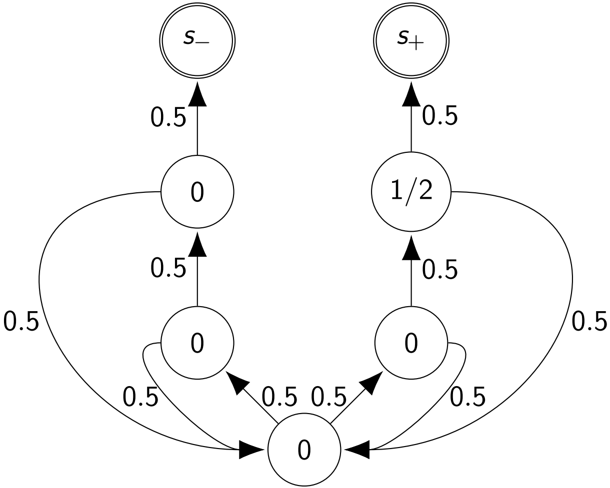
Moreover, if $v_0(s) = \mathbb{1}[s = s_+]$ and $v_{i+1} := \mathrm{Update}(v_i)$, then

$$(v_i) \xrightarrow[i \to \infty]{} \mathsf{val}$$

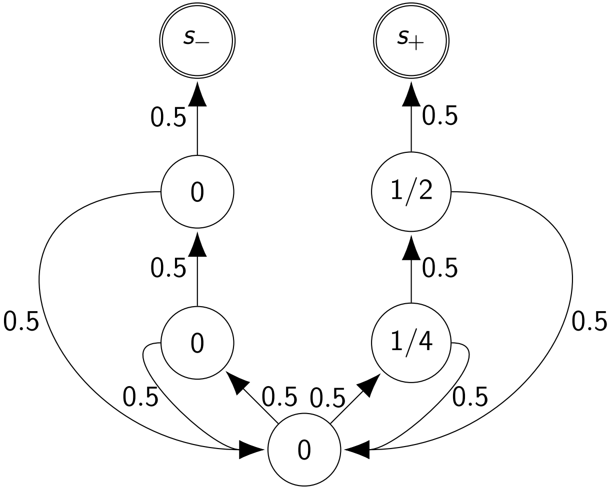
Value Iteration



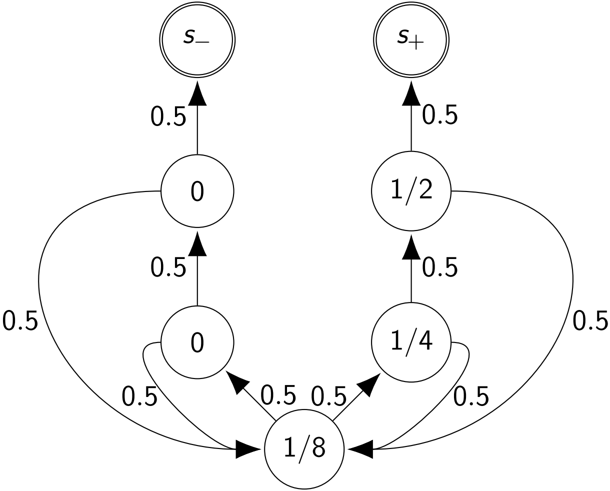
Value Iteration



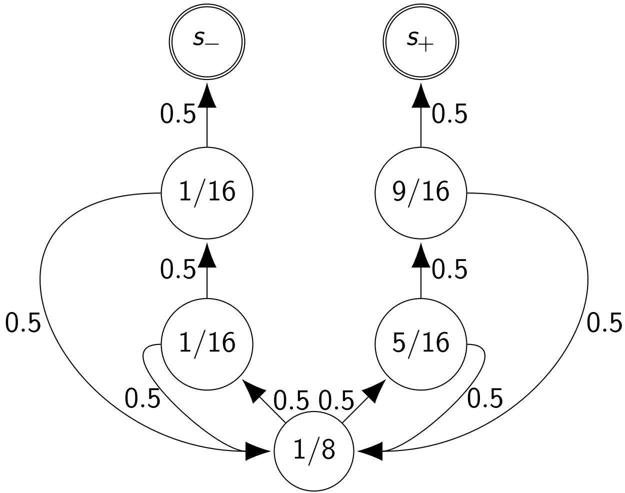
Value Iteration



Value Iteration



Value Iteration



Value Iteration speed

Lemma

The speed of convergence is driven by

$$\|v_{|S|i} - \mathrm{val}\|_\infty \leq \left(1 - p_{\min}^{|S|}\right)^i \|v_0 - \mathrm{val}\|_\infty$$

So, to have an ε*-approximation of* val, *Value Iteration requires many iterations:*

$$(-\log(\varepsilon)|S|/p_{\min})^{\Omega(|S|)}$$

In other words, **exponentially** *many.*

Value Iteration speed: Levels

Lemma

The speed of convergence is driven by

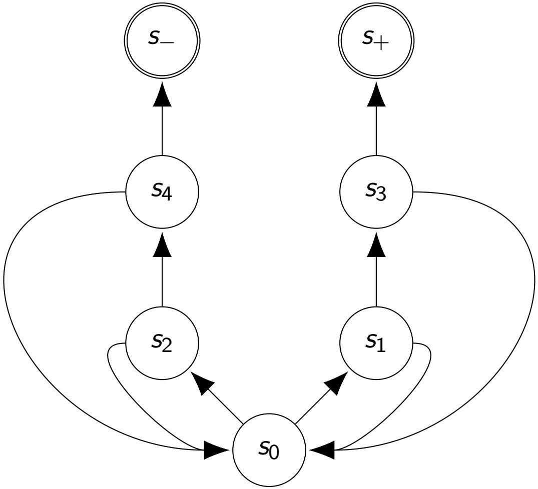
$$\|v_{Li} - \mathsf{val}\|_\infty \leq \left(1 - p_{\min}^L\right)^i \|v_0 - \mathsf{val}\|_\infty$$

where $L \leq |S|$ is the maximum graph distance to the target or sink. An ε-approximation of val *requires many iterations:*

$$(-\log(\varepsilon) L / p_{\min})^{\Omega(L)}$$

In other words, **exponentially** *many.*

Stochastic system



Value Iteration speed, improved

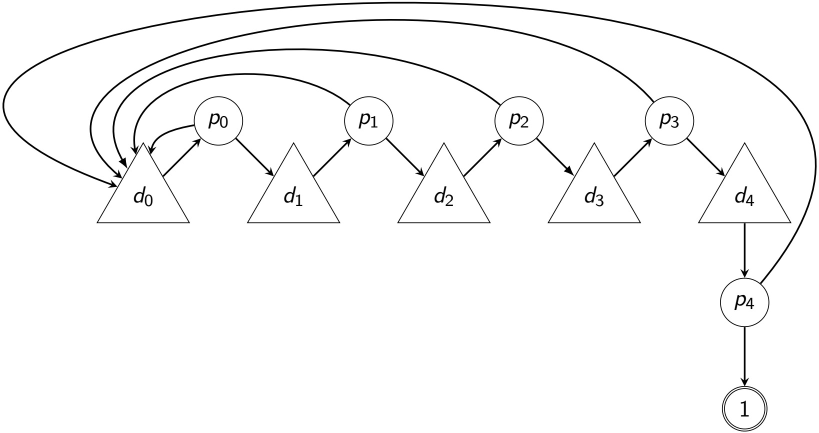
Theorem

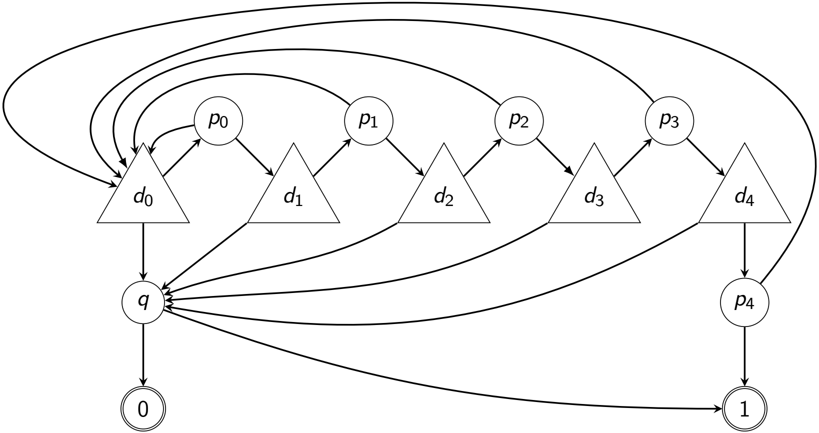
To have an ε-approximation of val*, we require fewer iterations:*

$$(-\log(\varepsilon)|S|/p_{\min})^{\mathcal{O}(\sqrt{|S|})}$$

In other words, **subexponentially** *many.*

MDPs


p_0
p_1
p_2
p_3
d_0
d_1
d_2
d_3
d_4
p_4
1


p_0
p_1
p_2
p_3
d_0
d_1
d_2
d_3
d_4
q
p_4
0
1

Value Iteration is not accelerated
by having low distance to the targets in MDPs.

What about practical speedups?

Practical implementation

Guessing, solving, and verifying implies a recursive algorithm.
Recursion depth is given by the number of states guessed.
Guessing more than 3 states is impractical.

Practical considerations:

- Early Verification of a Guess
- Reusing Bounds
- Picking the Guessed States
- Benefiting from VI

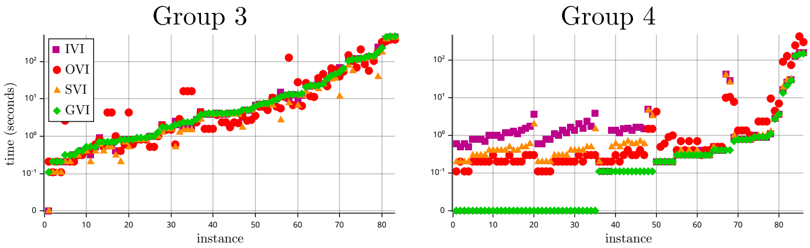
We consider the Quantitative Verification Benchmark Set (QVBS).
The following VI-based variants were considered.

- Interval VI
  Classic two-sided VI
- Optimistic VI
  Candidates for upper or lower bounds are proposed while Interval VI is running
- Sound VI
  Deduce upper bounds based on lower bounds
- Guessing VI
  Our algorithm

Grouping results

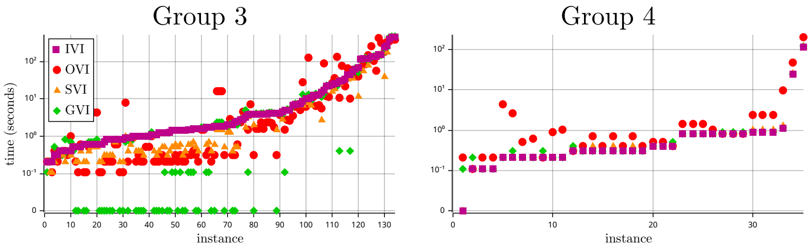
- Group 1
  All algorithms are fast, i.e., they take at most 0.1 seconds.
  170 instances.
- Group 2
  The fastest and slowest algorithms
  are only at most 1.10 times of each other.
  135 instances.
- Group 3
  Guessing VI is not the fastest approach.
  83 instances.
- Group 4
  All other instances not considered before.
  86 instances.

Performance: Guessing VI

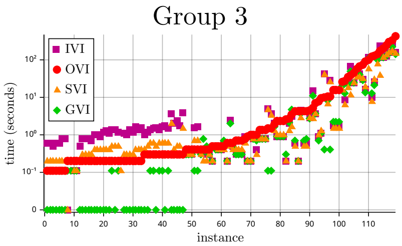


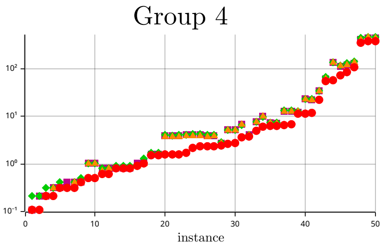
Time in seconds of all algorithms over instances in Groups 3 and 4 in increasing order **according to GVI** and displayed in logarithmic scale.

Performance: Interval VI



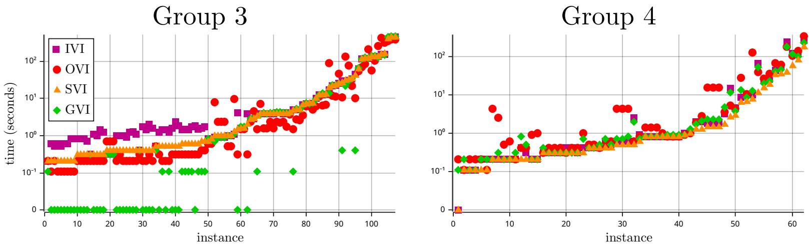
Time in seconds of all algorithms over instances in Groups 3 and 4 in increasing order **according to IVI** and displayed in logarithmic scale.





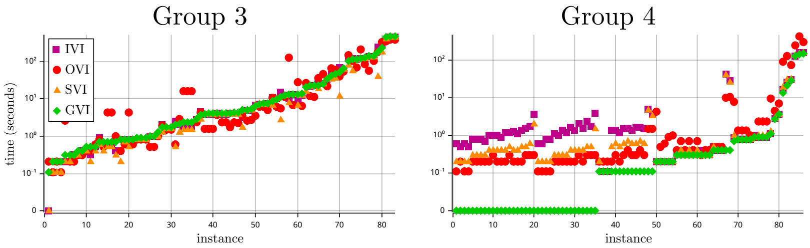
Time in seconds of all algorithms over instances in Groups 3 and 4 in increasing order **according to OVI** and displayed in logarithmic scale.

Performance: Sound VI



Time in seconds of all algorithms over instances in Groups 3 and 4 in increasing order **according to SVI** and displayed in logarithmic scale.

Performance: Guessing VI



Time in seconds of all algorithms over instances in Groups 3 and 4 in increasing order **according to GVI** and displayed in logarithmic scale.

Thank you!